Review of Linear Algebra

- A matrix is a rectangular array of numbers enclosed in brackets.
- Examples, (i) (0 1 2), (ii) $\begin{pmatrix} 0 & 1 \\ 5 & 7 \end{pmatrix}$, (iii) $\begin{pmatrix} \pi \\ 2 \\ 0.4 \end{pmatrix}$
- A general matrix may be written as, **A**= $\begin{pmatrix} a11 & \dots & a1m \\ . & \dots & . \\ an1 & \dots & anm \end{pmatrix}$
- This is a $n \times m$ matrix, in shorthand **A**= (a_{ij})
- Column vectors -> $\underline{\mathbf{a}}$, row vectors, $\bar{\mathbf{a}}$
- Thus, the matrix A=($\underline{\mathbf{a}}_1$, $\underline{\mathbf{a}}_2$,…, $\underline{\mathbf{a}}_m$)
- A diagonal matrix is, **A**= $\begin{pmatrix} a11 & \dots & 0 \\ . & ajj & . \\ 0 & \dots & amm \end{pmatrix}$, **I**= $\begin{pmatrix} 1 & \dots & 0 \\ . & 1 & . \\ 0 & \dots & 1 \end{pmatrix}$ (identity matrix)

Equality, Addition, Subtraction, Scalar Multiplication

- Addition and Subtraction
  A+**B**= $(a_{ij}+b_{ij})$; **A**-**B**= $(a_{ij}-b_{ij})$; **A**+**B**= **B**+**A**
- Examples: $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 1+5 & 2+6 \\ 3+7 & 4+8 \end{pmatrix}$
- Scalar multiplication, c**A**= (ca_{ij}); c(**A**+**B**)= c**A**+c**B**
- The transpose of a matrix is obtained by interchanging the rows and columns of a matrix, $\mathbf{A}^T$ is the transpose of **A**. If **A** is $n \times m$ then is $m \times n$.
- Example: $\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$

Matrix multiplication

- $\bar{a}\,\underline{b} = (a_1 b_1 + \cdots + a_n b_n)$, $\underline{b}\bar{a}$ is defined but quite different.
- **A** is ($n\times m$), **B** is ($m\times l$), **AB**= $\begin{pmatrix} \bar{a}_1 \\ . \\ \bar{a}_m \end{pmatrix}(\underline{b}_1..\underline{b}_l)=\begin{pmatrix} \bar{a}_1\underline{b}_1 & .. & \bar{a}_1\underline{b}_l \\ .. & .. & .. \\ \bar{a}_m\underline{b}_1 & .. & \bar{a}_m\underline{b}_l \end{pmatrix}$
- In general **AB**$\neq$**BA**
- Example, $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 1\cdot 1 + 2\cdot 4 \\ 3\cdot 1 + 4\cdot 4 \end{pmatrix} = \begin{pmatrix} 9 \\ 19 \end{pmatrix}$
- If A is $n\times m$, then $\mathbf{I}_n\mathbf{A}=\mathbf{A}$ and $\mathbf{A}\mathbf{I}_m=\mathbf{A}$
- If either $\bar{a}$ or $\underline{b}$ is filled with zeros then $\bar{a}\,\underline{b}=0$

Systems of Linear Equations

- $12x_1+3x_2=9$
  $2x_1+x_2=6$
- $\begin{pmatrix} 12 & 3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 9 \\ 6 \end{pmatrix}$
- General system of linear equations

$$a_{11}x_1 + \cdots + a_{1n}x_n = b_1$$

$$\cdots\cdot$$

$$a_{n1}x_1 + \cdots + a_{nn}x_n = b_n$$

- $\mathbf{A}\underline{\mathbf{x}}=\underline{\mathbf{b}}$
- An inverse function of **A**, $\mathbf{A}^{-1}$, satisfies $\mathbf{A}^{-1}\mathbf{A}=\mathbf{I}_n$
- $\mathbf{A}^{-1}\mathbf{A}\underline{\mathbf{x}}= \mathbf{A}^{-1}\underline{\mathbf{b}}$ -> $\mathbf{I}_n\underline{\mathbf{x}}= \mathbf{A}^{-1}\underline{\mathbf{b}}$ -> $\underline{\mathbf{x}}= \mathbf{A}^{-1}\underline{\mathbf{b}}$

Determinants

- Inverse and determinants are only defined for square matrices
- A determinant for a 2 by 2 is easy. Larger matrices can be done recursively but are tedious (see R function chol2inv).
- $det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21}$
- Useful facts
  1: $\mathbf{A}^{-1}$ exists if and only if det $\mathbf{A} \neq 0$
  2: det ($\mathbf{AB}$) = det $\mathbf{A}$ det $\mathbf{B}$
  3: det $\mathbf{A}^T$ = det $\mathbf{A}$
  4: There is a non-trivial solution to $\mathbf{A}\underline{\mathbf{x}}=\underline{\mathbf{0}}$ if and only if det $\mathbf{A}=0$
- Prove 4 by assuming det $\mathbf{A} \neq 0$ and then there exists and inverse and a trivial solution
  5: There is a unique solution to $\mathbf{A}\underline{\mathbf{x}}=\underline{\mathbf{b}}$ if and only if det $\mathbf{A} \neq 0$